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NUMERICAL APPROXIMATION OF SINGULAR, NON-SINGULAR CAPUTO DEFINITION AND COMPARING WITH EXACT SOLUTION

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Abstract: - Caputo definition of fractional derivative comes in two different kernels. The singular kernel and other is non-singular kernel for fractional derivative and fractional integration. Numerical approximation of these definitions are proposed and comparing the results with the exact solution. Semi-group property has been proved using the MATLAB simulation. A new approach of finding fractional derivative using the initial values proposed in terms of power series. Behavior and properties of the singular type kernel at non-singular points is investigated for the handling of singular points. General solution of fractional differential equation is discussed to illustrate the exact solution for the application of control system.

Keywords: - Singular and non-singular kernel, fractional derivative, singularity points, fractional control system, Fractional Numerical Method.

Introduction

Global approach of solving the system problem is motivated to use the fractional order methods. This method attracted the researchers to solve the open research problem in system modeling, simulation and control. Most real time data and initial values of systems satisfy and match with the solution of singular Caputo [1]. The Caputo Definition given as,

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t \frac{f'(\tau)\, d\tau}{(t-\tau)^{\alpha}} \tag{1}$$

Dynamical systems provide unique solution with bounded parameters and non-zero initial conditions. Non-zero condition is uncommon and method has to be developed to match the physical initial conditions with the fractional definition. Non-singular Caputo fractional derivative equation is as given [2],

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t e^{\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]} f'(\tau)\, d\tau \tag{2}$$

Replacing with exponential kernel at singular point the exponential component become unity (at t=τ). Non-singular kernel does not satisfy initial value conditions except α=1 and singular kernel is must to satisfy initial value conditions [3]. Some important remarks are,

Fractional derivative and integration at singular point does not exists.

If singular kernel replaced with non-singular kernel, then the initial value condition does not match as per the real physical initial values [4].

To meet real physical initial values, the kernel should be singular Non-singular fractional derivative (equation-5) is a weighted similarity with other definition given in equation (1).

Laplace domain transform is the Laplace transform of J_t^α f(t) (equation-6) which is convolution in Laplace domain (equation-7). Solving the convolution in Laplace domain is not desirable for control system due to the little influence on the performance of real time control loop. Complex area selection become too much complicated for control loop instead of avoiding the singular point which is more practicable then to increase real time computation burden.

$$J_t^{\alpha} f(t) = \frac{M(\alpha) e^{\left[-\frac{\alpha(t)}{1-\alpha}\right]}}{(1-\alpha)} \int_a^t e^{[-s(\tau)]} f(\tau)\, d\tau \tag{3}$$

$$L[\ J_t^{\alpha} f(t)] = \int_0^s Q(s-u)\, P(u) du \tag{4}$$

Where, $L[Q(\alpha,t)] = Q(s)$; $L\left[\int_a^t e^{[-s(\tau)]} f(\tau)\, d\tau\right] = P(s)$

$$Q(\alpha,t) = \frac{M(\alpha) e^{\left[-\frac{\alpha(t)}{1-\alpha}\right]}}{(1-\alpha)} \ ; -s = \frac{\alpha}{1-\alpha}$$

Preliminaries

Singular kernel is must and essential to meet the initial values condition and singular points should avoid in the computation of control loop problems. The singular point arises at t=τ in continuous form. Fractional numerical approximation of Caputo definition proposed in [3] given in equation (5). The singular point arises at each of n=k which avoided using mathematical technique during approximation of Caputo derivative.

$$y^{\propto}(f(x)) \approx \sum_{n=0}^{N} \left\{ \left[\frac{1}{h^{\propto}\Gamma(2-\alpha)}\right] \left[\left(y(x_n)\, G_{n,n}^{\alpha} - y(x_0)\, G_{n,1}^{\alpha}\ \right) + \left(\sum_{k=1}^{n-1} y(x_k)\, M_{n,k}^{\alpha}\right) \right] \right\}$$
(5)

Where, $M_{n,k}^{\alpha} = G_{n,k}^{\alpha} - G_{n,k+1}^{\alpha}$; $G_{n,k}^{\alpha} = (n-k+1)^{1-\alpha} - (n-k)^{1-\alpha}$

Theorem 1: - The generalize fractional derivative can be defined in-terms of power series as,

$$D^{\propto} f(t) = \frac{t^{m-\alpha-1}}{\Gamma(m-\alpha)} \left[\sum_{n=0}^{N} \frac{(-1)^n}{t^n\ \Gamma(n+1)} \sum_{k=0}^{n} (-1)^k \left\{\left[t^{n-k} f^{m-(1+k)}(t)\right] - \left[c^{n-k} f^{m-(1+k)}(c)\right]\right\} \prod_{i=1}^{n} [(m-\alpha-1)-(i-1)] \right] \tag{6}$$

Where, ‘c’ is initial value and f(c) is the initial function value. $m-1 < \alpha < m$, m is positive integer. Γ(.) = Euler’s Gamma Function. $\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1}\ dt$

Proof: - Consider, Fractional Caputo definition in generalized form as,

$$D^{\propto} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_c^t (t-\tau)^{m-\alpha-1} f^m(\tau)\, d\tau \tag{7}$$

Let, $g(x) = (t-x)^{m-\alpha-1} = t^{m-\alpha-1} \left(1-\frac{x}{t}\right)^{m-\alpha-1}$; using binomial expansion theorem.

$$g(x) = t^{m-\alpha-1} \left[\sum_{n=0}^{N} (-1)^n \frac{\left(\frac{x}{t}\right)^n}{n!} \prod_{i=1}^{n} [(m-\alpha-1)-(i-1)]\right] \tag{8}$$

Using value of (8) putting in (7),

$$D^{\propto} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_c^t \left\{ t^{m-\alpha-1} \left[\sum_{n=0}^{N} (-1)^n \frac{\left(\frac{\tau}{t}\right)^n}{n!} \prod_{i=1}^{n} [(m-\alpha-1)-(i-1)]\right]\right\} f^m(\tau)\, d\tau$$

$D^{\propto}f(t) = \frac{t^{m-\alpha-1}}{\Gamma(\mathrm{m}-\alpha)} \left[\sum_{n=0}^{N} \frac{(-1)^n}{t^n\ \Gamma(n+1)} \int_c^t \tau^n\, f^m(\tau)\, d\tau\ \prod_{i=1}^{n}[(m-\alpha-1)-(i-1)]\right]$
(9)

Let, $F(t) = \int_c^t \tau^n\, f^m(\tau)\, d\tau$ and finding the general solution,

$I_n = \int_c^t \tau^n\, f^m(\tau)\, d\tau = \sum_{k=0}^{n}(-1)^k \left\{\left[t^{n-k}\, f^{m-(1+k)}(t)\right] - \left[c^{n-k}\, f^{m-(1+k)}(c)\right]\right\}$
(10)

If $m = 0$ and $n = s - 1$,which convert (10) into Mellin transform with limit 0 to ∞
Using value of (10) putting in (9),

$D^{\propto}f(t) = \frac{t^{m-\alpha-1}}{\Gamma(\mathrm{m}-\alpha)} \left[\sum_{n=0}^{N} \frac{(-1)^n}{t^n\ \Gamma(n+1)} \sum_{k=0}^{n}(-1)^k \left\{\left[t^{n-k}\, f^{m-(1+k)}(t)\right] - \left[c^{n-k}\, f^{m-(1+k)}(c)\right]\right\} \prod_{i=1}^{n}[(m-\alpha-1)-(i-1)]\right]$

Hence, Proved.

Theorem 2: - At singular point ($t = \tau$), the first order (m=1) fractional derivative is given as,
$D^{\propto}f(t) = \frac{g}{\Gamma(1-\alpha)}\ [f(t) - f(c)]$; ‘c’ is initial value and f(c) is the initial function value; $g = f(p, \alpha)$ such that $\frac{1}{e} \leq g \leq e$;

Proof:-Put m=1 in equation (10), $D^{\propto}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t (t-\tau)^{-\alpha} f^1(\tau)\, d\tau$
(11)

At singularity $-\tau = 0$; Convolution is independent of first or second function time-folding. Let us replace the singular kernel with non-singular kernel as $t^{\alpha} = e^{x}$and solving approximate upto 3 terms,

$t^{\alpha} = e^{\left[-1+\sqrt{2t^{\alpha}-1}\right]}$ (12)

For $0 < \alpha < 1$ at $t = 0$ (at singular point only)

$g = \frac{1}{t^{\alpha}} = \frac{1}{e^{\left[-1+\sqrt{2t^{\alpha}-1}\right]}} = \frac{1}{e^{\left[-1+\sqrt{-1}\right]}} = \frac{1}{e^{[-1+\mathrm{i}]}} = \frac{1}{1/e} = e$

For $-1 < \alpha < 0$ at $t = 0$ (at singular point only)

$g = t^{\alpha} = e^{\left[-1+\sqrt{2t^{\alpha}-1}\right]} = e^{\left[-1+\sqrt{-1}\right]} = e^{[-1+\mathrm{i}]} = 1/e$

$g = [e^{-1}, e^{1}]$ hence $\frac{1}{e} \leq g \leq e$ and g= e^{α}

Rewriting equation (11) by using the identity derived for singular point,

$D^{\propto}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_c^t g\, f^1(\tau)\, d\tau$
(13)

After integrating equation (13), $D^{\propto}f(t) = \frac{g}{\Gamma(1-\alpha)}\ [f(t) - f(c)]$; and hence proved.

Theorem 3: - if $f = e^{p}$ then the value of ‘p’ is $p_{k+1} = log_e\left[\frac{e^{p_k}}{\Gamma(1-\alpha)}\right]$ which is recursive function. And p is bounded such as $-1 \leq p \leq 1$ and $\alpha \neq p$

Proof: - Consider Three cases at $= 0$, $\alpha = 1$ and $\alpha = -1$

At $\alpha = 0$; $D^0 f(t) = f(t)$; $\Gamma(1) = 1$

At $\alpha = 1$; $D^1 f(t) = f^1(t)$; $\Gamma(0) = \infty$

At $\alpha = -1$; $D^{-1} f(t) = \int f(t)dt$; $\Gamma(2) = 1$

Substituting the identity in equation (13),

At $\alpha = 0$; $p = log_e\left[\frac{f(t)}{[f(t)-f(c)]}\right]$
(14)

At $\alpha = 1$; $p = log_e \left[\frac{f^1(t)}{[f(t)-f(c)]}\right]$
(15)
At $\alpha = -1$; $p = log_e \left[\frac{\int f(t)dt}{[f(t)-f(c)]}\right]$
(16)
Using the general fractional operator,
$p = log_e \left[\frac{D^{\propto} f(t)}{[f(t)-f(c)]}\right]$
(17)
Using Theorem 2 substituting value of $D^{\propto} f(t)$ for singular point,
$p = log_e \left[\frac{e^p}{\Gamma(1-\alpha)}\right]$
(18)
Value of p obtained as, $p_{k+1} = log_e \left[\frac{e^{p_k}}{\Gamma(1-\alpha)}\right]$; hence proved.

Corollary 1: - A normalized function $N(\alpha) = 1 - \frac{1}{\Gamma(1-\alpha)}$ is exist for singular points. Such that $(c) = f(t)$; From equation (17), Figure 1 shows the normalized function $N(\alpha)$

$$f(c) = \frac{e^p f(t) - D^{\propto} f(t)}{e^p}$$

$$f(c) = \frac{e^p f(t) - \left(\frac{e^p}{\Gamma(1-\alpha)}[f(t)-f(c)]\right)}{e^p}$$

$$f(c) - \frac{f(c)}{\Gamma(1-\alpha)} = f(t) - \frac{f(t)}{\Gamma(1-\alpha)}$$

$$f(c)\left[1 - \frac{1}{\Gamma(1-\alpha)}\right] = f(t)\left[1 - \frac{1}{\Gamma(1-\alpha)}\right]$$

$N(\alpha) = 1 - \frac{1}{\Gamma(1-\alpha)}$ and $f(c) = f(t)$

Corollary 2: - At singular point the fractional derivative is $D^{\propto} f(t) = M'(\alpha)\,[f(t) - f(c)]$. Where, $M'(\alpha) = \frac{e^{\alpha}}{\Gamma(1-\alpha)}$; Figure 2 shows plot of $M'(\alpha)$ function.

Theorem 4: - Non-Singular kernel Caputo-Fabrizio definition.

$$D^{\alpha} f(x) \approx \left[\frac{M(\alpha)}{\alpha h}\right] \sum_{n=0}^{N} \left\{\left[y(x_n) MG_{n,n}^{\alpha,h} - y(x_0) MG_{n,1}^{\alpha,h}\right] + \left[\sum_{k=1}^{n-1} y(x_k)\, MGG_{n,k}^{\alpha,h}\right]\right\}$$
(19)

$MGG_{n,k}^{\alpha,h} = MG_{n,k}^{\alpha,h} - MG_{n,k+1}^{\alpha,h}$ and $MG_{n,k}^{\alpha,h} = e^{\left[\frac{\alpha h}{(1-\alpha)}(k-n)\right]} - e^{\left[\frac{\alpha h}{(1-\alpha)}(k-1-n)\right]}$

Proof :- From equation (2) and $x_n = nh$; $y_n = y_n(x_n) = y_n(nh)$; $x_k = kh$; $y'_k = y'_k(x_k) = y'_k(kh)$; a=0

$$\frac{(1-\alpha)}{M(\alpha)} D^{\alpha} f(x_n) \approx \int_0^t e^{\left[\frac{-\alpha(x_n - x)}{(1-\alpha)}\right]} f'(x)\, dx$$

$$\frac{(1-\alpha)}{M(\alpha)} D^{\alpha} f(x_n) \approx \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} e^{\left[\frac{-\alpha(nh-x)}{(1-\alpha)}\right]} f'(x_k)\, dx$$

Expansion of function $f'(x_k)$ approximated using Taylors method as,

$$f'(x_k) \cong \frac{y(x_k) - y(x_{k-1})}{h}$$

$\frac{(1-\alpha)}{M(\alpha)} D^{\alpha} f(x_n) \approx \sum_{k=1}^{n} \left[\frac{y(x_k) - y(x_{k-1})}{h}\right] \int_{x_{k-1}}^{x_k} e^{\left[\frac{-\alpha(nh-x)}{(1-\alpha)}\right]} dx$; after evaluation

$$D^{\alpha} f(x_n) \approx \left[\frac{M(\alpha)}{\alpha h}\right] \sum_{k=1}^{n} [y(x_k) - y(x_{k-1})] \left[e^{\left[\frac{\alpha h}{(1-\alpha)}(k-n)\right]} - e^{\left[\frac{\alpha h}{(1-\alpha)}(k-1-n)\right]}\right]$$

Let, $MG_{n,k}^{\alpha,h} = e^{\left[\frac{\alpha h}{(1-\alpha)}(k-n)\right]} - e^{\left[\frac{\alpha h}{(1-\alpha)}(k-1-n)\right]}$ and after simplification,
(20)

$$D^{\alpha} f(x_n) \approx \left[\frac{M(\alpha)}{\alpha h}\right] \left\{\left[y(x_n) MG_{n,n}^{\alpha,h} - y(x_0) MG_{n,1}^{\alpha,h}\right] + \left[\sum_{k=1}^{n-1} y(x_k)\left(MG_{n,k}^{\alpha,h} - MG_{n,k+1}^{\alpha,h}\right)\right]\right\}$$

Let, $MGG_{n,k}^{\alpha,h} = MG_{n,k}^{\alpha,h} - MG_{n,k+1}^{\alpha,h}$

(21)

$$D^{\alpha}f(x) \approx \left[\frac{M(\alpha)}{\alpha h}\right] \sum_{n=0}^{N}\{[y(x_n)MG_{n,n}^{\alpha,h} - y(x_0)MG_{n,1}^{\alpha,h}] + [\sum_{k=1}^{n-1} y(x_k)\, MGG_{n,k}^{\alpha,h}\,]\}$$

Corollary 3:- $MG_{n,k}^{0,h} = MG_{n,k}^{\alpha,0} = 0$; $MG_{n,n}^{\alpha,h} = MG_{k,k}^{\alpha,h} = 1 - e^{\left[-\frac{\alpha h}{(1-\alpha)}\right]}$; $MG_{n,n}^{1,h} = MG_{k,k}^{1,h} = 1$

$MG_{n,k}^{\alpha,1} = e^{\left[\frac{\alpha}{(1-\alpha)}\right]} - 1$ at $k - n = 1$; Series is convergent when,

$\int |y(x)\, MGG_{n,k}^{\alpha,h}|\, dx < \infty$ and $\int |y(x)MG_{n,n}^{\alpha,h} - y(x_0)MG_{n,1}^{\alpha,h}| < \infty$

The small perturbation ε is the relative error between two type of Caputo kernel given as,

$$\varepsilon = \frac{y(x_n)\left\{\left[\frac{1}{h^{\alpha}\Gamma(2-\alpha)}\right] - \left[\frac{M(\alpha)}{\alpha h}\left(1 - e^{\left[\frac{-\alpha h}{(1-\alpha)}\right]}\right)\right]\right\} + y(x_0)\left\{MG_{n,1}^{\alpha,h}\left[\frac{M(\alpha)}{\alpha h}\right] - G_{n,1}^{\alpha}\left[\frac{h^{\alpha}}{\Gamma(2-\alpha)}\right]\right\}}{\left[\frac{1}{h^{\alpha}\Gamma(2-\alpha)}\right]\left[\,(y(x_n) - y(x_0)\, G_{n,1}^{\alpha}\,)\,\right]}$$

(22)

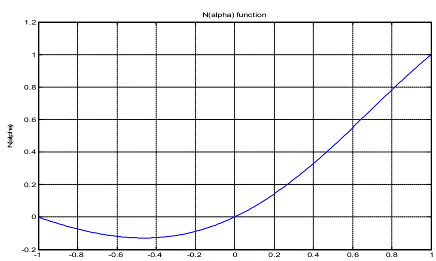
Where, $MG_{n,n}^{\alpha,h} = 1 - e^{\left[\frac{-\alpha h}{(1-\alpha)}\right]}$

(23)

Definition 1: - [5] Jorge and Juan corrected the fractional definition for the $0 < \alpha < 1$ as,

$$D_t^{\alpha} f(t) = \frac{1}{(1-\alpha)}\int_a^t e^{\left[-\frac{\alpha(t-x)}{1-\alpha}\right]} f'(x)\, dx$$

(24)



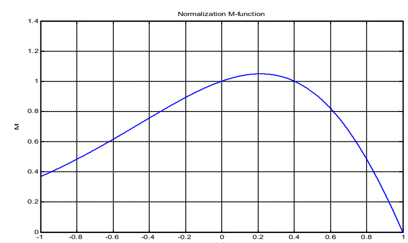


Figure 1 N-function **Figure 2 M'-Function**

1. Proposed Non-Singular Kernel

Non-Singular kernel fractional order derivative is proposed using the convolution integral transform [1].

Definition 2: - The Non-Singular Type kernel fractional derivative is given as,

$$D^{\alpha}f(t) = \frac{1}{\Gamma(p-\alpha)}\int_c^t (x)^{p-\alpha-1}\, f^p(t-x)\, dx \quad (25)$$

$p - 1 < \alpha < p$; p is positive integer; f^p = p^{th} order derivative of f; $f^p(0) \neq \infty$

Preposition: - At singular point $t = x$ and p=1

$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_c^t x^{-\alpha}\, f^1(0)\, dx$; and according to definition 2, $f^1(0) \neq \infty$, Thus the singularity avoided.

Property 1: - Using the convolution property the definition 2 holds the following identity,

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\int_c^t \frac{f'(\tau)\, d\tau}{(t-\tau)^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_c^t \frac{f'(t-\tau)\, d\tau}{(\tau)^{\alpha}} \quad (26)$$

Property 2: - Let, $f^1(0) = 1$ at singular point using preposition. Then the fractional derivative at singular point is,

$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (x)^{-\alpha}\, dx$; $D^{\alpha}f(t) = \frac{(t)^{1-\alpha}}{\Gamma(2-\alpha)}$

But, $D^{\alpha}(t) = \frac{(t)^{1-\alpha}}{\Gamma(2-\alpha)}$; Hence the fractional derivative can be defined in terms of initial values.

Property 3: - Proposed Non-Singular kernel is $\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)}$;

$\lim_{\alpha\to 0}\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} = u(x)$; $\lim_{\alpha\to 1}\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{1}{x}$; $\lim_{\alpha\to -1}\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} = x$; $\lim_{\alpha\to 0.5}\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{1}{\sqrt{\pi x}}$; $\lim_{\alpha\to -0.5}\frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} = \frac{\sqrt{\pi x}}{2}$

x is linear operator defined as,

$\alpha \to 0\,, x = 1 \quad ; \quad \alpha \to 1\,, \frac{1}{x} = \Delta t \quad ; \ \alpha \to -1\,, x = \int_0^x dt$

Thus, hold the following identity,

$\lim_{\alpha \to 0.5} \frac{(x)^{-\alpha}}{\Gamma(1-\alpha)} \lim_{\beta \to -0.5} \frac{(x)^{-\beta}}{\Gamma(1-\beta)} = \frac{1}{2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^x x^{5/2}\, dx$

The probability density function is proposed as,

$$p(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^t t^{5/2}\, dt \qquad (27)$$

Property 4: - Laplace Transform of proposed Non-singular kernel fractional definition.

$$L[D^{\propto} f(t)\} = \frac{sF(s)-f(0)}{(s)^{1-\alpha}}; \emptyset(s) = \frac{1}{(s)^{1-\alpha}}; \emptyset(t) = \frac{(t)^{-\alpha}}{\Gamma(1-\alpha)}; g(s) = \frac{sF(s)-f(0)}{(s)^{1-\alpha}}; \emptyset(s) = \frac{1}{(s)^{1-\beta}} \qquad (28)$$

$$L[D^{\beta}[D^{\propto} f(t)] = \frac{s^2F(s)-sf(0)-g(0)}{(s)^{2-(\beta+\alpha)}}; L[D^{\propto+\beta} f(t)] = \frac{s^2F(s)-sf(0)-g(0)}{(s)^{2-(\beta+\alpha)}} \qquad (29)$$

$$D^{\propto_1+\propto_2+\cdots+\propto_M} f(t) = \frac{1}{\Gamma((M-1)-(\propto_1+\propto_2+\cdots+\propto_M))} \int_0^t (\tau)^{(M-1)-(\propto_1+\propto_2+\cdots+\propto_M)} f^{(M-1)}(t-\tau)\, d\tau$$

(30)

$D^{\propto_1+\propto_2+\cdots+\propto_M} f(t) = D^{\propto_1} D^{\propto_2} \ldots D^{\propto_M} f(t)$

$0 < \propto_1, \propto_2, \ldots, \propto_M < 1$

Figure 3 shows the plot of semigroup property calculated using equation (5). The Mean Square Error (MSE) is 5.9548e-06.

Property 5: - Laplace Transform of Caputo-Fabrizio Non-Singular kernel fractional definition (Equation - 5),

$$L\,[D_t^\alpha f(t)] = \frac{M(\alpha)}{s(1-\alpha)+\alpha} \quad (s\,F(s) - f(0)) \qquad (31)$$

Fractional Order Integration of non-singular kernel is,

$$J_t^\alpha f(t) = \frac{M(\alpha)}{\alpha} \int_0^t e^{\left[-\frac{(1-\alpha)(t-j)}{\alpha}\right]} f(j)\, dj \qquad (32)$$

Laplace Transform of Fractional Order Integration of non-singular kernel is,

$$L\,[J_t^\alpha f(t)] = \left(\frac{M(\alpha)\,F(s)}{s\,\alpha+(1-\alpha)}\right) \qquad (33)$$

Let, $M(\alpha) = 1$ then,

$$L\,[D_t^\alpha f(t)] = \frac{1}{s(1-\alpha)+\alpha} \quad (s\,F(s) - f(0)) \qquad (34)$$

$$L\,[J_t^\alpha f(t)] = \frac{F(s)}{s\,\alpha+(1-\alpha)} \qquad (35)$$

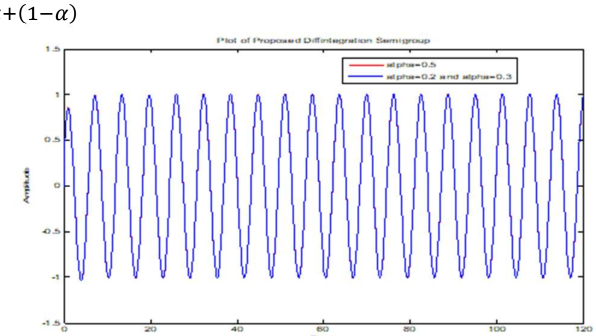


Figure 3:- Plot for showing semigroup property

Theorem 4: - The Fractional Derivative using initial values given as,

$$D^{\propto} f(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{f^n(0)\, n\, t^{n-\alpha}}{\Gamma(n+1)} \quad \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\,(k+n)} \prod_{i=1}^{k} (-\alpha - (i-1))\right]$$

(36)

$\alpha \neq R$; Where R is real positive integer number. ; $f^n = n^{th}$ order derivative of f; $f^n(0) \neq \infty$

Proof: - $D^{\propto} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f^1(\tau)\, d\tau$

Using Taylor series expansion,

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)x^n}{\Gamma(n+1)}$

$f^1(x) = \sum_{n=0}^{\infty} \frac{f^n(0)\, n\, x^{n-1}}{\Gamma(n+1)}$

$D^{\propto} f(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{f^n(0)\, n}{\Gamma(n+1)} \int_0^t (t-\tau)^{-\alpha}\, \tau^{n-1}\, d\tau$

$D^{\propto} f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{f^n(0)\, n}{\Gamma(n+1)} \int_0^t \left(1-\frac{\tau}{t}\right)^{-\alpha} \tau^{n-1}\, d\tau$

$D^{\propto} f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{f^n(0)\, n}{\Gamma(n+1)} \left[\sum_{k=0}^{\infty} \frac{(-1)^k\ t^{k+n}}{t^k\, \Gamma(k+1)\,(k+n)} \prod_{i=1}^{k}(-\alpha-(i-1))\right]$

$D^{\propto} f(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{f^n(0)\, n\, t^{n-\alpha}}{\Gamma(n+1)} \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\,(k+n)} \prod_{i=1}^{k}(-\alpha-(i-1))\right]$;

2. **Comparison with exact Solution**

Let, $f(t) = t^1$; $f'(t) = 1$ and all other higher derivatives are equal to zero.
Solution of f(t) using the present non-singular kernel (definition 2) is given as,

$$D_{0,t}^{\alpha}\,(t^1) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \quad (37)$$

The Caputo-Fabrizio Non-Singular kernel gives the following exact solution,

$$D_{0,t}^{\alpha}\,(t^1) = \frac{M(\alpha)}{\alpha}\left(1 - e^{\left[-\frac{\alpha t}{(1-\alpha)}\right]}\right) \quad (38)$$

Proposed non-singular kernel provides the useful solution exact matches with the classical definition of derivative and integration for $\alpha = 1$ and $\alpha = -1$ respectively. However, the Caputo-Fabrizio Non-singular kernel shows the mean square error (MSE) of 141.3798 and Jorge and Juan suggestion shows MSE of 702.5589 for classical type of integration (see table 1).

Table 1:- Numerical comparison with exact solution

MSE calculated using exact solution				
$\propto$	$f(t) = t$	Non-Singular Kernel (Proposed Method)	Caputo-Fabrizio Non-Singular Kernel	Jorge and Juan Suggestion
-1	$\frac{t^2}{2}$	0	141.3798	702.5589
0	t	0	NaN	NaN
1	1	0	1	0

Let, $f(t) = \sin\omega t$; $\alpha = 0.5$; $c = 0$; $\omega = 1$ and $f(t) = \sin t$; $f^1(t) = \cos t$
Solution using proposed non-singular kernel method is,

$D^{0.5} f(t) = \frac{1}{1.772453850905516} \int_0^t (\tau)^{-0.5} \cos(t-\tau)\, d\tau$

$D^{0.5} f(t) = \frac{1}{1.772453850905516} \int_0^t (\tau)^{-0.5} [\cos(t)\cos(\tau) + \sin(t)\sin(\tau)]\, d\tau$

$D^{0.5} f(t) = 0.5641895835477563\, \sqrt{2}\, \sqrt{\pi} \left[\left(\sin(t)\left(\left(S\left(\frac{\sqrt{2}\sqrt{t}}{\sqrt{\pi}}\right)\right) - (S(0))\right)\right) + \left(\cos(t)\left(\left(C\left(\frac{\sqrt{2}\sqrt{t}}{\sqrt{\pi}}\right)\right) - (C(0))\right)\right)\right] + c$

(39)

Where, The Fresnel Integral, $S(v) = \int \sin\left(\frac{\pi v^2}{2}\right) dv$; $C(v) = \int \cos\left(\frac{\pi v^2}{2}\right) dv$

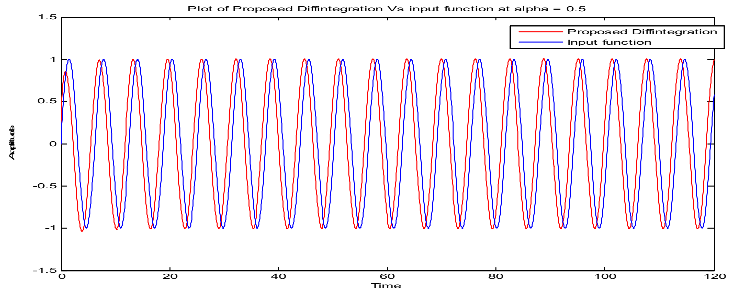


Figure 3:- Plot of proposed method and input function at alpha=0.5

$f(t) = \sin\omega t\ ;\ \alpha = -0.5\ ;\ c = 0\ ;\ \omega = 1\ ;\ f(t) = \sin t\ ;\ f^{1}(t) = \cos t$

$$D^{-0.5} f(t) = \frac{1}{0.886226925452758} \int_0^t (\tau)^{0.5} \cos(t-\tau)\, d\tau$$

$$D^{-0.5} f(t) = \frac{1}{0.886226925452758} \int_0^t (\tau)^{0.5} [\cos(t)\cos(\tau) + \sin(t)\sin(\tau)]\, d\tau$$

$$D^{-0.5} f(t) = 1.128379167095512\sqrt{t}\ \Big((\cos(t)\sin(t)) - \big(\sin(t)\ (\cos(t)-1)\big)\Big) - \frac{1.128379167095512\sqrt{\pi}\left[\left(\cos(t)\left(\left(S\left(\frac{\sqrt{2}\sqrt{t}}{\sqrt{\pi}}\right)\right)-(S(0))\right)\right)+\left(\sin(t)\left(\left(C\left(\frac{\sqrt{2}\sqrt{t}}{\sqrt{\pi}}\right)\right)-(C(0))\right)\right)\right]}{\sqrt{2}} \quad (40)$$

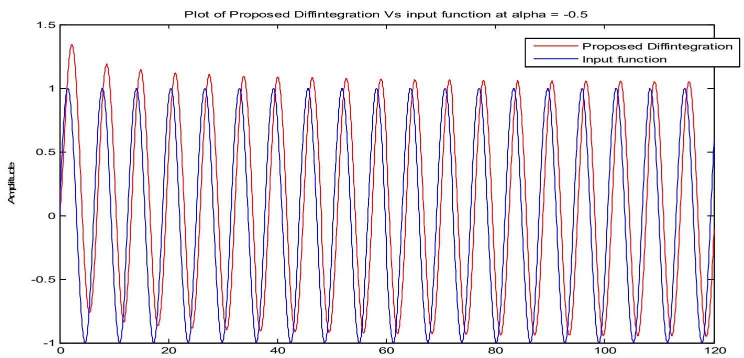


Figure 4:-Plot of proposed method and input function at alpha=-0.5

Solution of figure 3 and figure 4 is plotting using the numerical method given in equation (5). Output stabilization of proposed method is quick and responsive and hence suitable for the faster and dynamic systems.

3. General Solution of Fractional Differential Equation (FDE)

$D^{\alpha}_{0\ \text{to}\ t} f(t) + a\, f(t) = b\, u(t)$ and $L[D^{\alpha} f(t)\} = \frac{sF(s) - f(0)}{(s)^{1-\alpha}}$

(41)

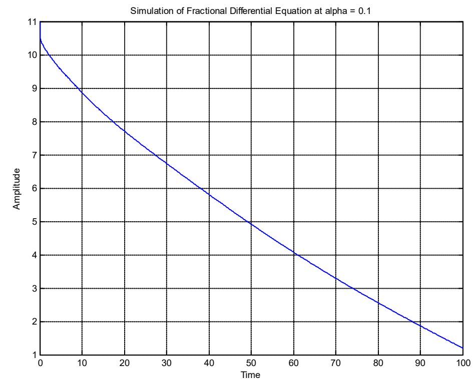
$$\frac{sF(s)-f(0)}{(s)^{1-\alpha}} + a\,F(s) = \frac{b}{s}$$

$$F(s) = \frac{b}{s\,((s)^{\alpha}+a)} + \frac{f(0)}{(s)^{1-\alpha}((s)^{\alpha}+a)} \quad (42)$$

$$f(t) = b\left[\frac{1}{a} - \frac{t^{\alpha-1} E_{\alpha,\alpha}(-a\,t^{\alpha})}{\frac{1}{a^{\alpha}}}\right] + f(0)\, E_{\alpha,1}(-a\,t^{\alpha}) \quad (43)$$

where, $E_{\alpha,\beta}(z)$ is Mettag-Leffler function.

Let, $\alpha = 1\ ;\ f(t) = \frac{b}{a}\left[1 - e^{(-at)}\right] + f(0)\, e^{(-at)}$. Satisfies the initial conditions of the system. Simulation of f(t) (equation 46) using the values $b = 10\ ;\ f(0) = 1;\ a = 1\ ;\ t \in [0,100]$ Observing the chaos at $\alpha = 1.5$ and the continuous ringing at $\alpha = 2$. However, it matches with the classical output at $\alpha = 1$.



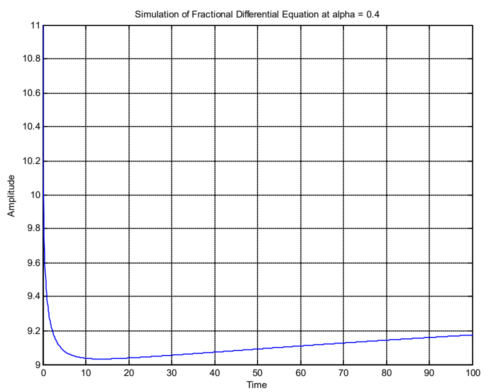
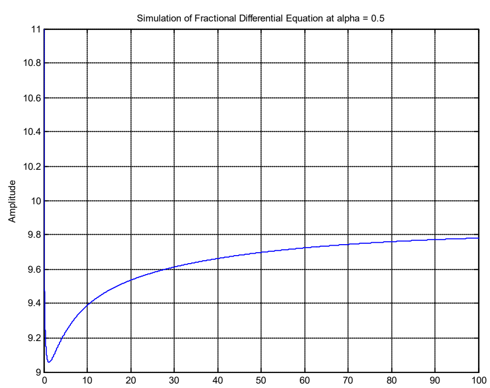


Figure 5: - Simulation of FDE at alpha=0.1

Figure 6: - Simulation of FDE at alpha=0.4



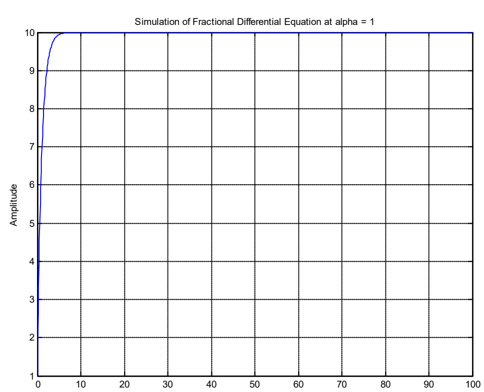
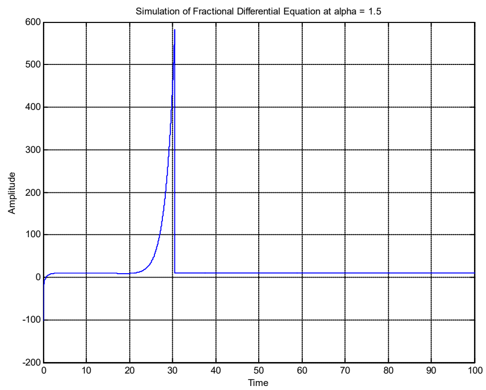


Figure 7: - Simulation of FDE at alpha=0.5

Figure 8: - Simulation of FDE at alpha=1



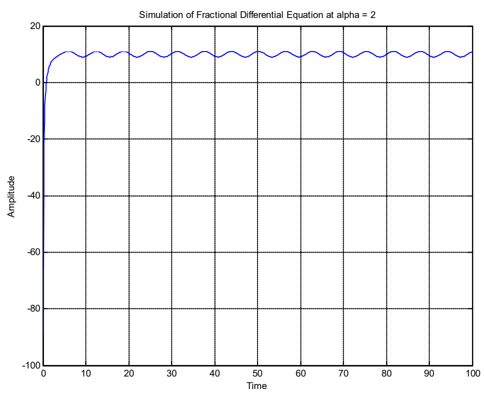


Figure 9: - Simulation of FDE at alpha=1.5 **Figure 10: - Simulation of FDE at alpha=2**

4. Application to Fractional Order Controller

FOPID Controller structure is given in equation (44),

$$y_c(t) = K_P\, e(t) + K_i\, D^{-\lambda}\, e(t) + K_d\, D^{\mu}\, e(t) \tag{44}$$

Property 1 is new method proposed and to calculate these fractional order derivative and integration the fractional numerical approximation equations are implemented in the MATLAB. Traditional PID controller is realized using the method mentioned in [6].

Consider a general fractional differential equation as,

$$u(t)b = K_P\, e(t) + K_i\, D^{-\lambda}\, e(t) + K_d\, D^{\mu}\, e(t) \tag{45}$$

Laplace transform of general fractional order differential equation is,

$$E(s) = \frac{s}{[K_P + K_i (s)^{-\lambda} + K_d (s)^{\mu}]} b + e(0)\left[\frac{[K_i]}{[K_P\,(s)^{\lambda+1} + K_i + K_d (s)^{\mu+\lambda}]} + \frac{[K_d]}{[K_P\,(s)^{1-\mu} + K_i (s)^{-\lambda-\mu} + K_d]}\right]$$

(46)

b and $e(0)$ is the transfer function parameters usually constant in nature. Controller algorithm is to minimize the error $E(s)$.

$$D^{-\lambda}\, e(t) \approx \sum_{n=0}^{N}\left\{\left[\frac{h^{\lambda}}{\Gamma(2+\lambda)}\right]\left[\left((e_n)\, G_{n,n}^{-\lambda} - y(e_0)\, G_{n,1}^{-\lambda}\right) + \left(\sum_{k=1}^{n-1}(e_k)\, M_{n,k}^{-\lambda}\right)\right]\right\}$$

(47)

The fractional derivative is implemented as,

$$D^{\mu}\, e(t) \approx \sum_{n=0}^{N}\left\{\left[\frac{1}{h^{\mu}\Gamma(2-\mu)}\right]\left[\left((e_n)\, G_{n,n}^{\mu} - (e_0)\, G_{n,1}^{\mu}\right) + \left(\sum_{k=1}^{n-1}(e_k)\, M_{n,k}^{\mu}\right)\right]\right\}$$

(48)

The Caputo-Fabrizio kernel fractional method is used to evaluate the fractional integration as,

$$D^{-\lambda}\, e(t) \approx \left[\frac{M(-\lambda)}{(-\lambda)h}\right] \sum_{n=0}^{N}\left\{\left[(e_n) MG_{n,n}^{-\lambda,h} - (e_0) MG_{n,1}^{-\lambda,h}\right] + \left[\sum_{k=1}^{n-1}(e_k)\, MGG_{n,k}^{-\lambda,h}\right]\right\} \tag{49}$$

On similar manner fractional derivative of Caputo-Fabrizio Non-Singular exponential type kernel as,

$$D^{\mu} e(t) \approx \left[\frac{M(\mu)}{\mu h}\right] \sum_{n=0}^{N} \left\{ \left[(e_n) MG_{n,n}^{\mu,h} - (e_0) MG_{n,1}^{\mu,h} \right] + \left[\sum_{k=1}^{n-1} (e_k)\ MGG_{n,k}^{\mu,h} \right] \right\} \tag{50}$$

Conclusion

Numerical method shows that the non-singular Caputo-Fabrizio method does not matches with the exact solution but the singular and proposed method matches with exact theoretical solution. Non-Singular Caputo-Fabrizio definition does not matches with the classical definition of derivative and integration shows the maximum mean square error. Singular point analysis has been showed that the fractional derivative and integration is dependent on the initial values of system and consistent with the initial values for f^n (0) nth order derivatives. Laplace analysis of the singular and non-singular type kernel showed that the non-singular kernel is a linear function in the "Laplace-domain" and thus useful in the application of control system.

Declaration of interests

"The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper".

Statement of Ethics

"The authors have no ethical conflicts to disclose"

Disclosure Statement

"The authors have no conflict of interest to declare"

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