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ADVANCED YANG-FOURIER TRANSFORMS TO LINEAR HEAT-CONDUCTION IN A SEMI-INFINITE FRACTAL BAR

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Abstract: - The main objective of the present paper is to solve the one-dimensional fractal heat-conduction problem in a semi-infinite fractal bar that has been developed by local fractional calculus employing the analytical Advanced Yang-Fourier transforms method.
Keywords: Advanced Yang-Fourier transforms, New special function i.e., the generalized S-function, Riemann-Liouville operator.

1. Introduction:

Advanced Yang-Fourier transforms, which the author obtained by generalising Yang-Fourier transforms, is a fractional calculus technique for resolving issues in mathematics, physics, and engineering. The use of fractional calculus has increased over the past 50 years [1-7]. Most of the fractional ordinary differential equations have exact analytic solutions, while others required either analytical approximations or numerical techniques to be applied, among them: fractional Fourier and Laplace transform [8,33], the heat-balance integral method [9-11], variation iteration method (VIM) [12-14], decomposition method [15,33], homotopy perturbation method [16,33], etc.

By using local fractional calculus theory to solve problems involving non-differential functions, the issues in fractal media can be effectively resolved [17-24]. Local fractional differential equations have been applied to model complex systems of fractal physical phenomena, [22-33] local fractional Fourier series method, [30], Yang-Fourier transform [31-33].

2. A New Generalized Special Function and Advanced Yang-Fourier transform and properties of Advanced Young -Fourier transform:

Here, we define a new generalized special function S as follows:

$$S=\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k\prod_{i=1}^{k}(a_i)^k a^k}{(b_1)_k\dots(b_q)_k k!\,\Gamma(1+\alpha)}z^k,\quad \alpha\in c, R(\alpha)>0. \qquad \dots (A)$$

After putting $\prod_{i=1}^{k}(a_i)^k=1$ in equation (A) the generalized S-function converts into the S-function [36].

After putting $a=1$ and $a_i=1$ in the above function (A), then the generalized S-function converts into the M-series [34].

And after putting $\frac{(a_1)_k \dots (a_p)_k \prod_{i=1}^{k} (a_i)^k a^k}{(b_1)_k \dots (b_q)_k} = 1$ in the generalized S-function (A), then the generalized S- function converts into the Mittag-Leffler function [35].

Let us Consider $f(x)$ is local fractional continuous in $(-\infty, \infty)$ we denote as $f(x) \in C_\alpha (-\infty, \infty)$ [24, 25, 27].

Let $f(x) \in C_\alpha (-\infty, \infty)$ The Advanced Yang-Fourier transform developed by authors written in the form [22, 23, 31, 32, 33]:

$$F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{\infty} S_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha \quad \dots (2.1)$$

After putting $\frac{(a_1)_k \dots (a_p)_k \prod_{i=1}^{k} (a_i)^k}{(b_1)_k \dots (b_q)_k k!} a^k = 1$, then it converts into the Yang-Fourier transform [33].

Then, the local fractional integration is given by [22-24, 27-29, 33]:

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b f(t) \, (dx)^\alpha = \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j) \, (\Delta t_j)^\alpha \quad \dots (2.2)$$

Where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = max\{\Delta t_1, \Delta t_2, \Delta t_j, \dots\} \{t_j, t_{j+1}\}$, $j = 0, \dots,$ $N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

If $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$, then its inversion formula takes the form [22, 23, 31, 32,33]

$$f(x) = F_\alpha^{-1}[f_\omega^{F,\alpha}(\omega)] = \frac{1}{\Gamma(\alpha+1)} \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} S_\alpha(-i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega) (d\omega)^\alpha \quad \dots (2.3)$$

After putting $\frac{(a_1)_k \dots (a_p)_k \prod_{i=1}^{k} (a_i)^k a^k}{(b_1)_k \dots (b_q)_k k!} = 1$ in, it converts into the Yang Inverse Fourier transform [33].

Some properties are shown as follows [22, 23]:

Let $F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega)$, and $F_\alpha\{g(x)\} = f_\omega^{F,\alpha}(\omega)$, and let be two constants. Then we have:

$$F_\alpha\{cf(x) + dg(x)\} = cF_\alpha\{f(x)\} + dF_\alpha\{g(x)\} \quad \dots (2.4)$$

If $\lim_{|x| \to \infty} f(x) = 0$, then we have:

$$F_\alpha\{f^\alpha(x)\} = i^\alpha \omega^\alpha F_\alpha\{f(x)\} \quad \dots (2.5)$$

In eq. (2.5) the local fractional derivative is defined as:

$$f^\alpha(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \quad \dots (2.6)$$

Where,

$$\Delta^\alpha [f(x) - f(x_0)] \cong \Gamma(1+\alpha) \Delta[f(x) - f(x_0)]$$

As a direct result, repeating this process, when:

$$f(0) = f^{\alpha}(0) = \cdots = f^{(k-1)\alpha,}(0) = 0$$

… (2.7)

$$F_{\alpha}\{f^{k\alpha}(x)\} = i^{\alpha}\omega^{\alpha}F_{\alpha}\{f(x)\}$$

… (2.8)

3. Heat conduction in a fractal semi-infinite

If a fractal body is subjected to a boundary perturbation, then the heat diffuses in-depth modeled by a constitutive relation where the rate of fractal heat flux $\overline{q}(x,y,z,t)$ is proportional to the local fractional gradient of the temperature [24,33], namely:

$$\overline{q}(x,y,z,t) = -K^{2\alpha}\nabla^{\alpha}T(x,y,z,t)$$

… (3.1)

Here the pre-factor K^{2a} is the thermal conductivity of the fractal material. Therefore, the fractal heat conduction equation without heat generation was suggested in [24] as:

$$K^{2\alpha}\frac{d^{2\alpha}T(x,y,z,t)}{dx^{2\alpha}} - \rho_{\alpha}c_{\alpha}\frac{d^{2\alpha}T(x,y,z,t)}{dx^{2\alpha}} = 0$$

… (3.2)

Where ρ_{α} and c_{α} are the density and the specific heat of the material, respectively.

The fractal heat-conduction equation with a volumetric heat generation $g(x,y,z,t)$ can be described as [24,33]:

$$K^{2\alpha}\nabla^{2\alpha}T(x,y,z,t) + g(x,y,z,t)\rho_{\alpha}c_{\alpha}\frac{\partial^{\alpha}T(x,y,z,t)}{\partial t^{\alpha}}$$

… (3.3)

The one-Dimensional fractal heat-conduction equation [24,33] reads as:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}} - \rho_{\alpha}c_{\alpha}\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}} = 0, \qquad 0 < x < \infty, t > 0$$

… (3.4)

with initial and boundary conditions are:

$$\frac{\partial^{\alpha}T(0,t)}{\partial t^{\alpha}} = S_{\alpha}t^{\alpha}, T(0,t) = 0$$

… (3.5)

The dimensionless forms of (3.4) and (3.5) are [27, 33]:

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}} = \frac{\partial^{\alpha}T(x,t)}{\partial x^{\alpha}} = 0$$

… (3.6)

$$\frac{\partial^{\alpha}T(0,t)}{\partial x^{\alpha}} = S_{\alpha}t^{\alpha}, T(0,t) = 0$$

… (3.7)

Based on eq. (3.4), the local fractional model for 1-D fractal heat-conduction in a fractal semi-infinite bar with a source term $g(x,t)$ is:

$$K^{2\alpha}\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}} - \rho_{\alpha}c_{\alpha}\frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}} = g(x,t), \qquad -\infty < x < \infty, t > 0$$

… (3.8)

With

$$T(x,0) = f(x), -\infty < x < \infty$$

… (3.9)

The dimensionless form of the model (3.8) and (3.9) is:

$$\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}} = \frac{\partial^{\alpha}T(x,t)}{\partial t^{\alpha}} = 0, \qquad -\infty < x < \infty, t > 0$$

… (3.10)

$$T(x,0) = f(x), -\infty < x < \infty$$

… (3.11)

4. Solutions by the Generalized New Yang-Fourier transform method:

Let us consider that

$$F_{\alpha}\{T(x,t)\} = T_{\omega}^{F,\alpha}(\omega,t)$$

is the Advanced Yang-Fourier transform of $T(x, t)$, regarded as a non-differentiable function of x. Applying the Yang-Fourier transform to the first term of Eq. (3.10), we obtain:

$$F_{\alpha}\left\{\frac{\partial^{2\alpha}T(x,t)}{\partial x^{2\alpha}}\right\} = (i^{2\alpha}\omega^{2\alpha})T_{\omega}^{F,\alpha}(\omega,t) = \omega^{2\alpha}T_{\omega}^{F,\alpha}(\omega,t)$$

… (4.1)

On the other hand, by changing the order of the local fractional differentiation and integration in the second term of eq. (3.10), we get:

$$F_{\alpha}\left\{\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}T(x,t)\right\} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}T_{\omega}^{F,\alpha}(\omega,t)$$

… (4.2)

For the initial value condition, the Yang-Fourier transform provides:

$$F_{\alpha}\{T(x,0)\} = T_{\omega}^{F,\alpha}(\omega,0) = F_{\alpha}\{f(x)\} = f_{\omega}^{F,\alpha}(\omega)$$

… (4.3)

Thus, we get from Eqn. (4.1), (4.2), and (4.3)

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}T_{\omega}^{F,\alpha}(\omega,t) + \omega^{2\alpha}T_{\omega}^{F,\alpha}(\omega,t) = 0, T_{\omega}^{F,\alpha}(\omega,0) = f_{\omega}^{F,\alpha}(\omega)$$

… (4.4)

This is an initial value problem of a local fractional differential equation with t as an independent variable and a parameter.

$$T(\omega,t) = f_{\omega}^{F,\alpha}(\omega)S_{\alpha}(-\omega^{2\alpha}t^{\alpha})$$

… (4.5)

Hence, using the inversion formula, eqn. (2.1), we get:

$$T(x,t) = \frac{1}{(2\pi)^{\alpha}}\int_{-\infty}^{\infty} S_{\alpha,}(i^{\alpha}\omega^{\alpha}x^{\alpha})f_{\omega}^{F,\alpha}(\omega)S_{\alpha,}(-\omega^{2\alpha}t^{\alpha})(d\omega)^{\alpha}$$

$$= (Mf)(x)$$

… (4.6)

$$M_{\omega}^{F,\alpha}(\omega) = \frac{1}{(2\pi)^{\alpha}} S_{\alpha,}(-\omega^{2\alpha} t^{\alpha})$$

… (4.7)

From [22, 24] we obtained,

$$F_{\alpha}\left\{S_{\alpha}\left(-\frac{\omega^{2\alpha}}{C^{2\alpha}}\right)\right\} = \frac{C^{\alpha}\pi^{\frac{\alpha}{2}}}{1.} \frac{1}{\Gamma(\alpha+1)} S_{\alpha,}\left(-\frac{C^{2\alpha}\omega^{2\alpha}}{4^{\alpha}}\right)$$

… (4.8)

Let $C^{2\alpha}/4^{\alpha} = t^{\alpha}$. Then we get:

$$F_{\alpha}\left\{S_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)\right\} = \frac{1}{\Gamma(\alpha+1)} \frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}{.} + S_{\alpha}(-\omega^{2\alpha}t^{\alpha}) = \frac{1}{\Gamma(\alpha+1)} \frac{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}}{.} (2\pi)^{\alpha} M_{\omega}^{F,\alpha}(\omega)$$

… (4.9)

Thus, $M_{\omega}^{F,\alpha}(\omega)$ have the inverse:

$$\frac{1}{(2\pi)^{\alpha}} \int_{-\infty}^{\infty} S_{\alpha}(i^{\alpha}\omega^{\alpha}x^{\alpha}) M_{\omega}^{F,\alpha,}(\omega)(d\omega)^{\alpha} = \frac{1}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}} \frac{1}{(2\pi)^{\alpha}} \Gamma(\alpha+1)\, S_{\alpha}\left(-\frac{\omega^{2\alpha}}{4^{\alpha}t^{\alpha}}\right)$$

… (4.10)

Hence, we get:

$$T(x,t) = (Mf)(x) = \frac{\Gamma(1+\alpha)}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\xi) S_{\alpha}\left(-\frac{(x-\xi)^{2\,\alpha}}{4^{\alpha}t^{\alpha}}\right)(d\xi)^{\alpha}$$

… (4.11)

The analysis is done now.

Special case:

After putting

$$\frac{(a_1)_k \ldots (a_p)_k \prod_{i=1}^{k}(a_i)^k a^k}{(b_1)_k \ldots (b_q)_k \mathrm{k}!} = 1$$

then the Generalized S- function converts into the Mittag-Leffler function and the solution of Advanced Yang Fourier Transforms converts into Yang Fourier Transforms results [33]

$$T(x,t) = (Mf)(x) = \frac{\Gamma(1+\alpha)}{4^{\alpha}t^{\frac{\alpha}{2}}\pi^{\frac{\alpha}{2}}} \int_{-\infty}^{\infty} f(\xi) E_{\alpha}\left(-\frac{(x-\xi)^{2\,\alpha}}{4^{\alpha}t^{\alpha}}\right)(d\xi)^{\alpha}$$

… (4.12)

5. Conclusions:

In this paper, we presented an analytical solution of 1-Dimensional linear heat conduction in the fractal semi-infinite bar by the Advanced Yang-Fourier transform of non-differentiable functions. We have applied a partial fractional differential equation on a Cantor set, which has led to the above results, which are very helpful in solving real-world problems.

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